Supplementary Material of “Estimation Based Adaptive Constraint Control for a Class of Coupled String Systems”
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I. LIST OF MATERIALS
This PDF file includes the following materials.
1) Contributions;
2) Dynamics Analysis;
3) Proof of Lemma 1;
4) Proof of Lemma 2;
5) Proof of Theorem 1;
6) Four Remarks;

II. CONTRIBUTIONS
The main contributions of this paper are listed below:
1) Compared to [16], the tension on the string system studied in this paper is a spatiotemporally varying function. The boundary tension of the string is constrained by applying the logarithmic BLF to ensure that the boundary tension remains within the constraint range \( |T(t)| \leq T_M \);
2) A common solution to deal with unknown boundary disturbances is to apply symbolic functions, which is a relatively simple method. However, since the symbolic function has discontinuity, the controller constructed based on it may have chattering phenomenon. Incorporating the Lyapunov function, two disturbance observers are designed to estimate unknown boundary disturbances in this paper, which avoid the chattering phenomenon induced by the sign function.
3) The case of unknown parameters of the string system is considered, and the adaptive method is utilized to compensate the uncertainty of system. Two adaptive boundary controllers are designed to effectively mitigate string vibrations.

III. DYNAMICS ANALYSIS
Analyzing the coupled string system from the dynamics perspective, the kinetic energy \( E_k(t) \) and the potential energy \( E_p(t) \) of the string system are expressed as
\[
E_k(t) = \frac{\rho}{2} \int_0^l \left[ x_1'(s,t)^2 + y_1'(s,t)^2 \right] ds \\
+ \frac{m}{2} \left[ x_2'(t)^2 + y_2'(t)^2 \right]
\]
and
\[
E_p(t) = \frac{1}{2} \int_0^l T(s, x_2(s,t), y_2(s,t)) x_2'(s,t) ds \\
+ \frac{EA}{2} \int_0^l \left[ y_2(s,t) + \frac{1}{2} x_2'(s,t)^2 \right] ds.
\]
The virtual work done by distributed disturbances \( f_x(s,t), f_y(s,t) \) on the string and boundary disturbances \( d_x(t), d_y(t) \) on the tip payload can be expressed as
\[
\delta W_f(t) = \int_0^l \left[ f_x(s,t) \delta x(s,t) + f_y(s,t) \delta y(s,t) \right] ds \\
+ d_x(t) \delta x(t) + d_y(t) \delta y(t).
\]
In order to restrain the vibrations, boundary control forces \( U_x(t), U_y(t) \) are imported at the boundary of the string. The virtual work done by the control is given by
\[
\delta W_m(t) = U_x(t) \delta x(t) + U_y(t) \delta y(t).
\]
Therefore, the total virtual work done on the system is described as
\[
\delta W(t) = \delta W_f(t) + \delta W_m(t).
\]

IV. PROOF OF LEMMA 1
Lemma 1: The upper and lower bounds of the Lyapunov function given by (16) are
\[
0 \leq \alpha_1 \left[ (\Pi(t) + \Pi_2(t)) + \overline{\nu}_0^2 \left( \ell, t \right) + \overline{m}^2(t) + \overline{EA}^2 \left( t \right) \right] \\
\leq \Gamma(t) \leq \alpha_2 \left[ (\Pi(t) + \Pi_2(t)) + \overline{\nu}_0^2 \left( \ell, t \right) + \overline{m}^2(t) + \overline{EA}^2 \left( t \right) \right]
\]
where \( \alpha_1, \alpha_2 > 0 \) and \( \Pi(t) = \int_0^l \left( \overline{x}_1^2 + \overline{y}_1^2 + \overline{x}_2^2 + \overline{y}_2^2 \right) ds \).
Proof: According to Young’s inequality and the inequality \( 2\overline{x}_1^2(t) \leq \overline{x}_2^2(t) \), we obtain
\[
-\frac{1}{2\sigma} \int_0^l \overline{x}_1^2 ds - \sigma \int_0^l \overline{x}_2^2 ds \\
\leq \frac{1}{\sigma} \int_0^l \overline{x}_2^2 ds + \sigma \int_0^l \overline{x}_1^2 ds
\]
where \( \sigma > 0 \) is a constant. Furthermore, by the definition of \( \Gamma_1(t) \), one has
\[
\frac{a}{2} \min \left[ \rho, \frac{EA}{2\sigma}, \kappa + EA \left( \frac{1}{4} + \sigma \right) \right] \Pi(t) \leq \Gamma_1(t)
\]
and
\[
\frac{a}{2} \max \left[ \rho, \frac{EA}{2\sigma}, \kappa + EA \left( \frac{1}{4} + \sigma \right) \right] \Pi(t)
\]
Thus, we have
\[
0 \leq \omega_1 \Pi(t) \leq \Gamma_1(t) \leq \omega_2 \Pi(t)
\]
where \( \omega_1 = (a/2) \min \left[ \rho, \frac{EA}{2\sigma}, \kappa + EA \left( \frac{1}{4} + \sigma \right) \right] \geq 0 \), \( \omega_2 = (a/2) \max \left[ \rho, \frac{EA}{2\sigma}, \kappa + EA \left( \frac{1}{4} + \sigma \right) \right] > 0 \).

Using Young’s inequality for (19), one obtains
\[
\left| \Gamma_3(t) \right| \leq A\rho \int_0^l \left( \overline{x}_1^2 + \overline{x}_2^2 + \overline{y}_1^2 + \overline{y}_2^2 \right) ds \leq \theta_1 \Pi(t)
\]
where \( \theta_1 = A\rho \), thus (11) is equivalent to
\[
-\theta_1 \Pi(t) \leq \Gamma_3(t) \leq \theta_1 \Pi(t)
\]

V. PROOF OF LEMMA 2
Lemma 2: The time derivative of (16) is upper bounded, i.e.,
\[
\Gamma_4(t) \leq -\alpha \Gamma(t) + \varepsilon
\]
where \( \alpha, \varepsilon > 0 \).
Proof: The derivative of \( \Gamma_1(t) \) along time is
\[
\Gamma_1(t) = \alpha_0 \int_0^t x_1 x_3 ds + \alpha_1 \int_0^t y_1 y_3 ds + \alpha_2 \int_0^t T_0(s) x_3 x_5 ds \\
+ aE \int_0^t (y_1 + \frac{1}{2} x_2^2) (y_3 + x_3 x_5) ds \\
+ 2a \int_0^t \kappa(s) x_3^2 x_5 ds.
\]

(16)

The system governing equations are substituted into the above equality, it results that
\[
\Gamma_1(t) = a \int_0^t T_0(s) (x_3 x_3 + x_5 x_5) ds + a \int_0^t T_0(s) x_5 x_5 ds \\
+ 2a \int_0^t \kappa(s) \frac{x_3^2 x_3 + 3x_3^2 x_5}{2} ds + a_E \int_0^t x_3^2 x_5 ds \\
+ 2a \int_0^t \kappa(s) x_3 x_5 ds + a_E \int_0^t x_3 x_5 ds \\
+ aE \int_0^t y_1 x_3 + x_3 x_5 ds + a_E \int_0^t x_3 y_5 ds \\
+ aE \int_0^t y_1 (y_3 + x_5 x_5) ds + a \int_0^t x_3 f_2 ds \\
+ aE \int_0^t (y_3 x_3 + x_5 x_5) ds + a \int_0^t y_3 f_2 ds.
\]

(17)

Integrating by parts and applying Young’s inequality, one has
\[
\Gamma_1(t) \leq a \sigma_1 \xi(t) E_\infty x_3(x_3 + x_5) + T_0(t) x_5(x_5) \\
+ a_E \int_0^t x_3^2 ds + 2a \int_0^t \kappa(t) x_3^2 ds + a \sigma_1 \int_0^t x_3^2 ds \\
+ a \sigma_2 \int_0^t y_3^2 ds + a \sigma_1 \int_0^t y_3^2 ds \\
+ a \int_0^t x_3 f_2 ds + a \int_0^t y_3 f_2 ds.
\]

(18)

where \( \sigma_1, \sigma_2 > 0 \) are constants.

The time derivative of \( \Gamma_2(t) \) is
\[
\Gamma_2(t) = a \zeta(t) \left[ U_1(t) + d_1(t) - T_0(t) x_3(t) - 2a \xi(t) x_3(t) \\
- E_\infty x_3(t) y_3(t) - E_\infty \frac{2b}{b^2 - x_3^2(t)} \ln \frac{b - x_3(t)}{b^2 - x_3^2(t)} \\
+ am \zeta(t) x_3(t) y_3(t) - a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds
\]

(19)

Combing the auxiliary signal (7), (8) with the boundary conditions (4), (5), one gets
\[
\Gamma_2(t) = a \zeta(t) \left[ U_1(t) + d_1(t) - T_0(t) x_3(t) - 2a \xi(t) x_3(t) \\
- E_\infty x_3(t) y_3(t) - E_\infty \frac{2b}{b^2 - x_3^2(t)} \ln \frac{b - x_3(t)}{b^2 - x_3^2(t)} \\
+ am \zeta(t) x_3(t) y_3(t) - a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
+ a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds
\]

(20)

One the basis of the controllers (9), (10) and disturbance observers (11), (12), it holds that
\[
\Gamma_2(t) = -ak_1 \zeta(t) \ln \frac{x_3(t)}{b^2 - x_3^2(t)} - ak_2 x_3(t) x_5(t) \\
- ak_2 x_3^2(t) + am \zeta(t) x_3(t) y_3(t) - a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
- a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds \\
- a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds
\]

(21)

By utilizing Young’s inequality yields
\[
\Gamma_2(t) \leq -ak_1 \zeta(t) \ln \frac{x_3(t)}{b^2 - x_3^2(t)} - ak_2 x_3^2(t) + \frac{ak_2}{2} x_3^2(t) \\
- ak_2 x_3^2(t) - \frac{ak_4}{2} x_3^2(t) + \frac{ak_4}{2} x_3^2(t) \\
- a \sigma_3 \int_0^t d_3(t) d_3(t) ds + a \sigma_3 \int_0^t d_3(t) d_3(t) ds
\]

(22)

Differentiating \( \Gamma_3(t) \) in time is
\[
\Gamma_3(t) = \alpha \int_0^t x_3 x_5 ds + \alpha \int_0^t x_3 x_5 ds \\
+ \alpha \int_0^t y_5 y_3 ds + \alpha \int_0^t y_5 y_3 ds \\
= B_2(t) + B_2(t) + B_2(t) + B_2(t)
\]

(23)

where \( B_2(t), B_2(t), B_2(t) \) and \( B_2(t) \) will be calculated in the following, respectively.

For \( B_2(t) = \lambda_2 \int_0^t x_3 x_5 ds \), substituting (1) into it, we obtain
\[
B_2(t) = \lambda \int_0^t x_3 x_5 ds + \lambda \int_0^t x_3 x_5 ds \\
+ 2 \lambda \int_0^t x_3 x_5 ds + 6 \lambda \int_0^t x_3 x_5 ds \\
+ \lambda \int_0^t x_3 x_5 ds + \lambda \int_0^t x_3 x_5 ds \\
+ \lambda \int_0^t x_3 x_5 ds + \lambda \int_0^t x_3 x_5 ds
\]

(24)

Using integration by parts, one has
\[
B_1(t) = \lambda \int_0^t x_3^2(t) + \lambda \int_0^t x_3^2(t) y_3(t)
\]
\[
- \frac{\lambda E}{2} \int_0^T \dot{x}_2^2 \phi_1(t) dt - \frac{\lambda}{2} \int_0^T \dot{T}_0^2(s) - T_{0r}(s) ds \int_0^T x_2^2 ds \\
- \frac{3}{2} \dot{x}_1 \int_0^T \phi_1(t) ds - \frac{3 \lambda E}{8} \int_0^T x_1^4 ds \\
+ \frac{\lambda E}{2} \int_0^T \dot{x}_2^2 \phi_2(t) ds + \lambda \int_0^T x_2^2 \phi_3(t) ds \\
+ \frac{3 \lambda}{2} \dot{x}_1 \int_0^T \phi_2(t) ds + \frac{3 \lambda E}{8} \int_0^T x_1^4 ds.
\]

For \( B_2(t) = \alpha \int_0^T x_2(t) \phi_1(t) dt \) and \( B_4(t) = \alpha \int_0^T x_4(t) \phi_2(t) dt \), integrating by parts, one gets
\[
B_2(t) = \frac{\alpha \lambda E}{2} \int_0^T x_2(t) \phi_1(t) dt - \frac{\lambda E}{2} \int_0^T x_2^2 \phi_1(t) dt \\
B_4(t) = \frac{\alpha \lambda E}{2} \int_0^T x_4(t) \phi_2(t) dt - \frac{\lambda E}{2} \int_0^T x_2^2 \phi_2(t) dt.
\]

For \( B_3(t) = \alpha \int_0^T x_3(t) \phi_1(t) dt \), from the governing equation (2), it has
\[
B_3(t) = \lambda E \int_0^T x_3(t) \phi_1(t) dt + \alpha \int_0^T x_2(t) \phi_1(t) dt \\
+ \lambda \int_0^T x_2^2 \phi_1(t) dt.
\]

Further, one yields
\[
B_5(t) = \frac{\lambda E}{2} \int_0^T x_5(t) \phi_1(t) dt - \frac{\lambda E}{2} \int_0^T x_2^2 \phi_1(t) dt \\
+ \frac{3 \lambda E}{8} \int_0^T x_1^4 \phi_1(t) dt + \frac{\lambda E}{2} \int_0^T x_2^2 \phi_2(t) dt \\
+ \lambda \int_0^T x_2^2 \phi_3(t) dt.
\]

Substituting (25)–(27) and (29) into (23), one has
\[
\Gamma_3(t) \leq \frac{\lambda E}{2} x_5^2(t) + \frac{3 \lambda E}{8} x_1^4(t) + \frac{\lambda E}{2} x_2^2(t) \\
+ \frac{\lambda E}{2} x_2^2(t) y_2(t) - \frac{\lambda}{2} \int_0^T x_2^2 \phi_1(t) dt \\
- \frac{\lambda}{2} \int_0^T \phi_1(t) ds - T_{0r}(s) - T_{0r}(s) - T_0^2(s) - T_0^2(s) \\
+ \frac{3 \lambda}{2} x_1^2(t) \phi_1(t) ds + \frac{\lambda E}{2} \int_0^T x_2^2 \phi_2(t) dt \\
+ \frac{\lambda E}{2} \int_0^T x_2^2 \phi_3(t) dt - \lambda \int_0^T x_2^2 \phi_2(t) dt.
\]

where \( \sigma_3, \sigma_4, \sigma_5 > 0 \) are constants.

The time derivative of \( \Gamma_4(t) \) is
\[
\Gamma_4(t) = -\gamma_3 \int_0^T \phi_1(t) \dot{T}_0(t) + \gamma_3 \int_0^T \phi_1(t) \dot{m}_t(t)
\]

Substituting the adaptive laws (13)–(15) into (31), one obtains
\[
\Gamma_4(t) = a \int_0^T \dot{T}_0(t) x_3(t) \phi_1(t) ds \\
- a \int_0^T \dot{T}_0(t) \phi_1(t) ds \\
- a \int_0^T \phi_1(t) x_3(t) \phi_1(t) ds + \alpha \int_0^T \phi_1(t) m_t(t)
\]

Further, utilizing Young's inequality, one gets
\[
\Gamma_4(t) \leq \frac{a \lambda E}{2} \int_0^T x_5(t) \phi_1(t) ds + \alpha \int_0^T x_2(t) \phi_1(t) dt \\
- \frac{\lambda E}{2} x_5^2(t) + \frac{3 \lambda E}{8} x_1^4(t) + \frac{\lambda E}{2} x_2^2(t) + \frac{\lambda E}{2} x_2^2(t) y_2(t) \\
- \frac{\lambda}{2} \int_0^T x_2^2 \phi_1(t) dt - \frac{\lambda}{2} \int_0^T \phi_1(t) ds - T_{0r}(s) - T_{0r}(s) - T_0^2(s) - T_0^2(s) \\
+ \frac{3 \lambda}{2} x_1^2(t) \phi_1(t) ds + \frac{\lambda E}{2} \int_0^T x_2^2 \phi_2(t) dt \\
+ \frac{\lambda E}{2} \int_0^T x_2^2 \phi_3(t) dt - \lambda \int_0^T x_2^2 \phi_2(t) dt.
\]
The above equation is equivalent to

\[ \Gamma_i(t) \leq -\alpha \Gamma_i(t) + e^{\alpha t}. \]

The above equation is equivalent to

\[ \frac{\partial}{\partial t} \left( \Gamma_i(t) e^{\alpha t} \right) \leq e^{\alpha t}. \]

Then, integrating (52) with respect to \( t \) from 0 to \( t \)

\[ \Gamma(t) \leq \left( \Gamma(0) - \frac{E}{\alpha} \right) e^{-\alpha t} + \frac{E}{\alpha} \leq \frac{\Gamma(0)}{e^{\alpha t}} + \frac{E}{\alpha} \]

which infers \( \Gamma(t) \) is bounded. Applying Young’s inequality and combining (17) with (21), we obtain

\[ \frac{aT_0}{2t} e^2(x+s,t) \leq \frac{aT_0}{2t} T_0(s) x^2(s,t) ds \]

\[ \leq \Gamma_1(t) \leq \Gamma_1(t) + \Gamma_2(t) \leq \frac{1}{\alpha_1} \Gamma(t) \]

\[ \frac{aE}{2t} e^2(x+s,t) \leq \frac{aE}{2t} \int_0^t e^2(x,t) ds \]

\[ \leq \Gamma_1(t) \leq \Gamma_1(t) + \Gamma_2(t) \leq \frac{1}{\alpha_1} \Gamma(t). \]

Rearrange the terms in the above inequality appropriately. It is obtained that \( x(s,t) \) and \( y(s,t) \) are bounded, that is

\[ |x(s,t)| \leq \sqrt{\frac{2t}{a^2 \alpha T_0}} \left( V(0) + \frac{E}{\alpha} \right) \]

and

\[ |y(s,t)| \leq \sqrt{\frac{2t}{a^2 \alpha E A}} \left( V(0) + \frac{E}{\alpha} \right) \]

which gives

\[ \lim_{t \to \infty} |x(s,t)| \leq \sqrt{\frac{2t}{a^2 \alpha T_0}} \]

\[ \lim_{t \to \infty} |y(s,t)| \leq \sqrt{\frac{2t}{a^2 \alpha E A}} \]

\[ \forall (s,t) \in [0,\ell] \times [0,\infty). \]

From the two inequalities (54), (55), we see that \( \Gamma_1(t) \) is bounded \( \forall t \in [0,\infty) \). Since \( \Gamma_1(t) \) is bounded, \( x_s(s,t), y_s(s,t), y_s(s,t) \) and \( y_s(s,t) \) are bounded \( \forall (s,t) \in [0,\ell] \times [0,\infty) \). From (1), the kinetic energy of the system is bounded, it follows that \( x_s(s,t) \) and \( y_s(s,t) \) are bounded by utilizing property 1. Similarly, it follows from (2) and property 1 that \( x_s(s,t) \) and \( y_s(s,t) \) are also bounded. Then, assuming Assumption 1, system governing equations, by means of boundary conditions and the above analysis, it is easy to obtain that \( x_s(s,t) \) and \( y_s(s,t) \) are also bounded. In addition, From (53), we have parameter estimation errors \( T_0(l,t), m(t), \) and \( E_A(t) \) are bounded. Thus, \( T_0(l,t), m(t), \) and \( E_A(t) \) are bounded. In conclusion, adaptive boundary control controllers \( U_s(t) \) and \( U_t(t) \) are designed and bounded. In summary, the proposed two adaptive boundary controllers \( U_s(t) \) and \( U_t(t) \) guarantee that all signals in the closed-loop system are bounded.

From the definition of \( \Gamma_2(t) \), it is easy to see that \( \Gamma_2(t) \to 0 \) as \( x_s(l,t) \to b \). By (53), we know that \( \Gamma_2(t) \) is bounded, so \( x_s(l,t) \neq b \). Considering \( b < x_s(l,t) < b \), further deduce that \( b < x_s(l,t) < b \), \( \forall t \in [0,\infty) \). Together with the tension expression, it is clear that \( -M < T(l,t) < T_M \) holds on \( [0,\infty) \); so the boundary tension \( T(l,t) \) satisfies the constraint.

\[ \text{VII. FOUR REMARKS} \]

Remark 1: The main tool utilized in this paper is the estimation-based adaptive constraint control method. Since this paper considers the case where the system has unknown boundary perturbations and unknown system parameters, two disturbance observers and three parameter adaptive laws are designed to estimate known boundary perturbations \( d_3(t) \), \( d_4(t) \), and unknown parameters \( T_0(l), m, E_A \), respectively. In addition, the boundary tension constraint problem of the system is also considered and the logarithmic BLF is selected to deal with it. Therefore, this paper effectively solves the vibration suppression problem of the transverse-longitudinal coupled string system using the estimation-based adaptive constraint control method.

Remark 2: In this paper, the unknown parameters \( T_0(l), m \) and \( E_A \) are considered. To solve this problem, the modification terms \( \phi_1, \phi_2, \) and \( \phi_3 \) in (13)–(15) are introduced to improve the robustness of the closed-loop system, which are used to regulate \( T_0(l,t), m(t), E_A(t) \), respectively, to avoid their fluctuation to very large values that may affect the control scheme.

Remark 3: From the above analysis, it can be seen that the system states \( x(s,t) \) and \( y(s,t) \) can be arbitrarily small as long as the design control parameters are properly chosen. According to the expression of \( \alpha_3 \), it is clear that the increase of control gains \( k_1, k_2 \) may cause \( \alpha_3 \) to increase. Then, the value of \( \alpha \) will increase, which eventually makes \( \sqrt{2\alpha E a M} \) decrease, i.e., it can give a better vibration reduction performance. But, it will generate a high gain control scheme by increasing \( k_1, k_2 \). Therefore, in practical engineering, to achieve good vibration reduction performance and to obtain an optimized control scheme, the design parameters should be carefully adapted.

Remark 4: All signals of the adaptive boundary controllers (9) and (10) can be obtained by backward difference algorithm or by sensor measurements. \( x(l,t) \) and \( y(l,t) \) are measured by laser displacement sensor at the boundary of the string and \( x_s(l,t) \) and \( y_s(l,t) \) are obtained by inclinometer. Furthermore, \( x_s(l,t), y_s(l,t), x_s(l,t), \) and \( y_s(l,t) \) are calculated using the backward difference algorithm for \( (x(s,t), y(l,t), x_s(l,t), y_s(l,t)) \), respectively.