Supplementary Material of "Estimation Based Adaptive Constraint Control for a Class of Coupled String Systems"

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I. List of Materials

This PDF file includes the following materials.
1) Contributions;
2) Dynamics Analysis;
3) Proof of Lemma 1;
4) Proof of Lemma 2;
5) Proof of Theorem 1;
6) Four Remarks;

II. Contributions

The main contributions of this paper are listed below:

1) Compared to [16], the tension on the string system studied in this paper is a spatiotemporally varying function. The boundary tension of the string is constrained by applying the logarithmic BLF to ensure that the boundary tension remains within the constraint range $|T(\ell,t)| \le T_M$.

2) A common solution to deal with unknown boundary disturbances is to apply symbolic functions, which is a relatively simple method. However, since the symbolic function has discontinuity, the controller constructed based on it may have chattering phenomenon. Incorporating the Lyapunov function, two disturbance observers are designed to estimate unknown boundary disturbances in this paper, which avoid the chattering phenomenon induced by the sign function.

3) The case of unknown parameters of the string system is considered, and the adaptive method is utilized to compensate the uncertainty of system. Two adaptive boundary controllers are designed to effectively mitigate string vibrations.

III. Dynamics Analysis

Analyzing the coupled string system from the dynamics perspective, the kinetic energy $E_k(t)$ and the potential energy $E_p(t)$ of the string system are expressed as

$$E_k(t) = \frac{\rho}{2}\int_0^\ell \left[x_t^2(s,t) + y_t^2(s,t)\right]ds + \frac{m}{2}\left[x_t^2(\ell,t) + y_t^2(\ell,t)\right] \tag{1}$$

$$E_p(t) = \frac{1}{2}\int_0^\ell T(s,x_s(s,t))\,x_s^2(s,t)\,ds + \frac{EA}{2}\int_0^\ell \left[y_s(s,t) + \frac{1}{2}x_s^2(s,t)\right]^2 ds. \tag{2}$$

The virtual work done by distributed disturbances $f_x(s,t)$, $f_y(s,t)$ on the string and boundary disturbances $d_x(t)$, $d_y(t)$ on the tip payload can be expressed as

$$\delta W_f(t) = \int_0^\ell \left[f_x(s,t)\delta x(s,t) + f_y(s,t)\delta y(s,t)\right]ds + d_x(t)\delta x(\ell,t) + d_y(t)\delta y(\ell,t). \tag{3}$$

In order to restrain the vibrations, boundary control forces $U_x(t)$, $U_y(t)$ are imported at the boundary of the string. The virtual work done by the control is given by

$$\delta W_m(t) = U_x(t)\delta x(\ell,t) + U_y(t)\delta y(\ell,t). \tag{4}$$

Therefore, the total virtual work done on the system is described as

$$\delta W(t) = \delta W_f(t) + \delta W_m(t) \tag{5}$$

Through the application of Hamilton's principle $\int_{t_1}^{t_2}\delta(E_k - E_p + W)dt = 0$, $t_1 < t < t_2$, the governing equations (1), (2) and boundary conditions (3)−(5) of the coupled string system are obtained after a series of lengthy and straightforward calculations.

IV. Proof of Lemma 1

Lemma 1: The upper and lower bounds of the Lyapunov function given by (16) are

$$0 \le \alpha_1\left[\Pi(t) + \Gamma_2(t) + \widetilde{T_0}^2(\ell,t) + \widetilde{m}^2(t) + \widetilde{EA}^2(t)\right] \le \Gamma(t) \le \alpha_2\left[\Pi(t) + \Gamma_2(t) + \widetilde{T_0}^2(\ell,t) + \widetilde{m}^2(t) + \widetilde{EA}^2(t)\right] \tag{6}$$

where $\alpha_1, \alpha_2 > 0$ and $\Pi(t) = \int_0^\ell \left(x_t^2 + y_t^2 + x_s^2 + y_s^2 + x_s^4\right)ds$.

Proof: According to Young's inequality and the inequality $2y_s^2(s,t) \le x_s^2(s,t)$, we obtain

$$-\frac{1}{2\sigma}\int_0^\ell x_s^2 ds - \sigma\int_0^\ell x_s^4 ds \le \int_0^\ell y_s x_s^2 ds \le \frac{1}{\sigma}\int_0^\ell y_s^2 ds + \sigma\int_0^\ell x_s^4 ds \tag{7}$$

where $\sigma > 0$ is a constant. Furthermore, by the definition of $\Gamma_1(t)$, one has

$$\frac{a}{2}\min\left[\rho, \underline{T_0} - \frac{EA}{2\sigma}, EA, \underline{\kappa} + EA\left(\frac{1}{4} - \sigma\right)\right]\Pi(t) \le \Gamma_1(t) \le \frac{a}{2}\max\left[\rho, \overline{T_0}, EA\left(1 + \frac{1}{\sigma}\right), \overline{\kappa} + EA\left(\frac{1}{4} + \sigma\right)\right]\Pi(t) \tag{8}$$

where σ satisfies

$$\begin{cases} T_0(\ell) > 0 \\ T_0(\ell) - \dfrac{EA}{2\sigma} > 0 \\ \kappa(\ell) + EA\left(\dfrac{1}{4} - \sigma\right) > 0 \\ \kappa(\ell) + EA\left(\dfrac{1}{4} + \sigma\right) > 0. \end{cases} \tag{9}$$

Thus, we have

$$0 \le \omega_1\Pi(t) \le \Gamma_1(t) \le \omega_2\Pi(t) \tag{10}$$

where $\omega_1 = (a/2)\min\left[\rho,\ \underline{T_0} - EA/2\sigma,\ EA,\ \underline{\kappa} + EA((1/4) - \sigma)\right] > 0$, $\omega_2 = (a/2)\max\left[\rho,\ \overline{T_0},\ \overline{\kappa} + EA(1 + 1/\sigma), EA(1/4 + \sigma)\right] > 0$.

Using Young's inequality for (19), one obtains

$$|\Gamma_3(t)| \le \lambda\rho\ell\int_0^\ell \left(x_t^2 + x_s^2 + y_t^2 + y_s^2\right)ds \le \theta_1\Pi(t) \tag{11}$$

where $\theta_1 = \lambda\rho\ell$, thus (11) is equivalent to

$$-\theta_1\Pi(t) \le \Gamma_3(t) \le \theta_1\Pi(t). \tag{12}$$

Accounting for λ is a small positive weighting constant that satisfies $0 < \lambda < \omega_1/\rho\ell$, thereby $0 < \theta_1 < \omega_1$, and

$$\begin{cases} \theta_2 = \omega_1 - \theta_1 \ge 0 \\ \theta_3 = \omega_2 + \theta_1 \ge 0. \end{cases} \tag{13}$$

Combining (10), (12) with (13), one gets

$$0 \le \theta_2\Pi(t) \le \Gamma_1(t) + \Gamma_3(t) \le \theta_3\Pi(t). \tag{14}$$

Based on (16), one obtains (1), where $\alpha_1 = \min(\theta_2, 1, \gamma_3/2, d\ \gamma_4/2, \gamma_5/2) > 0$, $\alpha_2 = \max(\theta_3, 1, \gamma_3/2, \gamma_4/2, \gamma_5/2) > 0$. ■

V. Proof of Lemma 2

Lemma 2: The time derivative of (16) is upper bounded, i.e.,

$$\Gamma_t(t) \le -\alpha\Gamma(t) + \varepsilon \tag{15}$$

where $\alpha, \varepsilon > 0$.

Proof: The derivative of $\Gamma_1(t)$ along time is

$$\Gamma_{1t}(t) = a\rho\int_0^\ell x_t x_{tt} ds + a\rho\int_0^\ell y_t y_{tt} ds + a\int_0^\ell T_0(s) x_s x_{st} ds$$
$$+ aEA\int_0^\ell \left(y_s + \frac{1}{2}x_s^2\right)(y_{st} + x_s x_{st}) ds$$
$$+ 2a\int_0^\ell \kappa(s) x_s^3 x_{st} ds. \quad (16)$$

The system governing equations are substituted into the above equality, it results that

$$\Gamma_{1t}(t) = a\int_0^\ell T_0(s)(x_t x_{ss} + x_s x_{st}) ds + a\int_0^\ell T_{0s}(s) x_t x_s ds$$
$$+ 2a\int_0^\ell \kappa(s)\left(x_s^3 x_{st} + 3x_t x_s^2 x_{ss}\right) ds + \frac{aEA}{2}\int_0^\ell x_s^3 x_{st} ds$$
$$+ 2a\int_0^\ell \kappa_s(s) x_t x_s^3 ds + \frac{3aEA}{2}\int_0^\ell x_t x_s^2 x_{ss} ds$$
$$+ aEA\int_0^\ell x_t(x_{ss} y_s + x_s y_{ss}) ds + \frac{aEA}{2}\int_0^\ell x_s^2 y_{st} ds$$
$$+ aEA\int_0^\ell y_t(y_{ss} + x_s x_{ss}) ds + a\int_0^\ell x_t f_x ds$$
$$+ aEA\int_0^\ell (y_s y_{st} + x_s x_{st}) ds + a\int_0^\ell y_t f_y ds. \quad (17)$$

Integrating by parts and applying Young's inequality, one has

$$\Gamma_{1t}(t) \le a x_t(\ell,t)\left[EA x_s(\ell,t) y_s(\ell,t) + T_0(\ell) x_s(\ell,t)\right.$$
$$\left. + \frac{EA}{2} x_s^3(\ell,t) + 2\kappa(\ell) x_s^3(\ell,t)\right] + a\sigma_1\int_0^\ell x_t^2 ds$$
$$+ a\sigma_2\int_0^\ell y_t^2 ds + a y_t(\ell,t)\left[EA y_s(\ell,t) + \frac{EA}{2} x_s^2(\ell,t)\right]$$
$$+ \frac{a}{\sigma_1}\int_0^\ell f_x^2 ds + \frac{a}{\sigma_2}\int_0^\ell f_y^2 ds \quad (18)$$

where $\sigma_1,\ \sigma_2 > 0$ are constants.

The time derivative of $\Gamma_2(t)$ is

$$\Gamma_{2t}(t) = am\zeta(t)\zeta_t(t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} + am\eta(t)\eta_t(t)$$
$$+ \frac{am}{2}\zeta^2(t)\frac{2x_s(\ell,t)x_{st}(\ell,t)}{b^2 - x_s^2(\ell,t)} + \frac{a}{\delta_1}\widetilde{d_x}(t)\widetilde{d_{xt}}(t)$$
$$+ \frac{a}{\delta_2}\widetilde{d_y}(t)\widetilde{d_{yt}}(t). \quad (19)$$

Combing the auxiliary signal (7), (8) with the boundary conditions (4), (5), one gets

$$\Gamma_{2t}(t) = a\zeta(t)\left[U_x(t) + d_x(t) - T_0(\ell)x_s(\ell,t) - 2\kappa(\ell)x_s^3(\ell,t)\right.$$
$$\left. - EA x_s(\ell,t) y_s(\ell,t) - \frac{EA}{2} x_s^3(\ell,t)\right]\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)}$$
$$+ am\zeta(t) x_{st}(\ell,t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} - \frac{a}{\delta_1}\widetilde{d_x}(t)\widehat{d_{xt}}(t)$$
$$+ a\eta(t)\left[U_y(t) + d_y(t) - EA y_s(\ell,t) - \frac{EA}{2} x_s^2(\ell,t)\right]$$
$$- \frac{a}{\delta_2}\widetilde{d_y}(t)\widehat{d_{yt}}(t) + \frac{a}{\delta_1}\widetilde{d_x}(t) d_{xt}(t) + \frac{a}{\delta_2}\widetilde{d_y}(t) d_{yt}(t)$$
$$+ am\zeta^2(t)\frac{x_s(\ell,t)x_{st}(\ell,t)}{b^2 - x_s^2(\ell,t)} + am\eta(t) y_{st}(\ell,t). \quad (20)$$

One the basis of the controllers (9), (10) and disturbance observers (11), (12), it holds that

$$\Gamma_{2t}(t) = -ak_1\zeta^2(t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} - ak_2 x_s(\ell,t) x_t(\ell,t)$$
$$- ak_2 x_t^2(\ell,t) + a\widetilde{m}(t)\zeta(t) x_{st}(\ell,t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)}$$
$$- a\zeta(t)\left[\widetilde{T_0}(\ell,t) x_s(\ell,t) + \widetilde{EA}(t) x_s(\ell,t) y_s(\ell,t)\right.$$
$$\left. + \frac{\widetilde{EA}(t)}{2} x_s^3(\ell,t)\right]\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} + a\gamma_1\widetilde{d_x}(t)(t)\widehat{d_x}(t)$$
$$+ a\widetilde{m}(t)\zeta^2(t)\frac{x_s(\ell,t)x_{st}(\ell,t)}{b^2 - x_s^2(\ell,t)} + a\widetilde{m}(t)\eta(t) y_{st}(\ell,t)$$
$$- a\zeta(t)\left[\widehat{T_0}(\ell,t) x_s(\ell,t) + \widehat{EA}(t) x_s(\ell,t) y_s(\ell,t)\right.$$
$$\left. + \frac{\widehat{EA}(t)}{2} x_s^3(\ell,t)\right] - ak_4 y_t^2(\ell,t) - 2a\kappa(\ell) x_s^3(\ell,t)\zeta(t)$$
$$- ak_4 y_s(\ell,t) y_t(\ell,t) + a\gamma_2\widetilde{d_y}(t)\widehat{d_y}(t) - ak_3\eta^2(t)$$
$$- a\eta(t)\left[EA y_s(\ell,t) + \frac{EA}{2} x_s^2(\ell,t)\right] + \frac{a}{\delta_1}\widetilde{d_x}(t) d_{xt}(t)$$
$$+ \frac{a}{\delta_2}\widetilde{d_y}(t) d_{yt}(t). \quad (21)$$

By utilizing Young's inequality yields

$$\Gamma_{2t}(t) \le -ak_1\zeta^2(t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} - ak_3\eta^2(t) + \frac{ak_2}{2} x_s^2(\ell,t)$$
$$- \frac{ak_2}{2} x_t^2(\ell,t) - \frac{ak_4}{2} y_t^2(\ell,t) + \frac{ak_4}{2} y_s^2(\ell,t)$$
$$- a\zeta(t)\left[\widetilde{T_0}(\ell,t) x_s(\ell,t) + \widetilde{EA}(t) x_s(\ell,t) y_s(\ell,t)\right.$$
$$\left. + \frac{\widetilde{EA}(t)}{2} x_s^3(\ell,t)\right]\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} - 2a\kappa(\ell) x_s^4(\ell,t)$$
$$- a\zeta(t)\left[\widehat{T_0}(\ell,t) x_s(\ell,t) + \widehat{EA}(t) x_s(\ell,t) y_s(\ell,t)\right.$$
$$\left. + \frac{\widehat{EA}(t)}{2} x_s^3(\ell,t)\right] + a\widetilde{m}(t)\zeta^2(t)\frac{x_s(\ell,t)x_{st}(\ell,t)}{b^2 - x_s^2(\ell,t)}$$
$$+ a\widetilde{m}(t)\zeta(t) x_{st}(\ell,t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} + \frac{a\gamma_1}{2} d_x^2(t)$$
$$- a\eta(t)\left[EA y_s(\ell,t) + \frac{EA}{2} x_s^2(\ell,t)\right] + \frac{a}{\delta_1\sigma_3} d_{xt}^2(t)$$
$$+ a\widetilde{m}(t)\eta(t) y_{st}(\ell,t) - \left(\frac{a\gamma_1}{2} - \frac{a\sigma_3}{\delta_1}\right)\widetilde{d_x}^2(t)$$
$$- 2a\kappa(\ell) x_s^3(\ell,t) x_t(\ell,t) + \frac{a\gamma_2}{2} d_y^2(t) + \frac{a}{\delta_2\sigma_4} d_{yt}^2(t)$$
$$- \left(\frac{a\gamma_2}{2} - \frac{a\sigma_4}{\delta_2}\right)\widetilde{d_y}^2(t). \quad (22)$$

Differentiating $\Gamma_3(t)$ in time is

$$\Gamma_{3t}(t) = \lambda\rho\int_0^\ell s x_{tt} x_s ds + \lambda\rho\int_0^\ell s x_t x_{st} ds$$
$$+ \lambda\rho\int_0^\ell s y_{tt} y_s ds + \lambda\rho\int_0^\ell s y_t y_{st} ds$$
$$= B_1(t) + B_2(t) + B_3(t) + B_4(t) \quad (23)$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$ and $B_4(t)$ will be calculated in the following, respectively.

For $B_1(t) = \lambda\rho\int_0^\ell s x_{tt} x_s ds$, substituting (1) into it, we obtain

$$B_1(t) = \lambda\int_0^\ell s T_0(s) x_s x_{ss} ds + \lambda\int_0^\ell s T_{0s}(s) x_s^2 ds$$
$$+ 2\lambda\int_0^\ell s\kappa_s(s) x_s^4 ds + 6\lambda\int_0^\ell s\kappa(s) x_s^3 x_{ss} ds$$
$$+ \lambda EA\int_0^\ell s x_s x_{ss} y_s ds + \lambda EA\int_0^\ell s x_s^2 y_{ss} ds$$
$$+ \frac{3\lambda EA}{2}\int_0^\ell s x_s^3 x_{ss} ds + \lambda\int_0^\ell s x_s f_x ds. \quad (24)$$

Using integration by parts, one has

$$B_1(t) = \frac{\lambda\ell T_0(\ell)}{2} x_s^2(\ell,t) + \frac{\lambda EA\ell}{2} x_s^2(\ell,t) y_s(\ell,t)$$

$$
\begin{aligned}
&-\frac{\lambda EA}{2}\int_0^\ell x_s^2 y_s ds-\frac{\lambda}{2}\int_0^\ell [T_0(s)-T_{0s}(s)]x_s^2 ds \\
&-\int_0^\ell\left[\frac{3\lambda}{2}\kappa(s)-\frac{\lambda}{2}s\kappa_s(s)+\frac{3\lambda EA}{8}\right]x_s^4 ds \\
&+\frac{\lambda EA}{2}\int_0^\ell s x_s^2 y_{ss} ds+\lambda\int_0^\ell s x_s f_x ds \\
&+\left[\frac{3\lambda\ell}{2}\kappa(\ell)+\frac{3\lambda\ell EA}{8}\right]x_s^4(\ell,t).
\end{aligned}
\tag{25}
$$

For $B_2(t)=\lambda\rho\int_0^\ell s x_t x_{st} ds$ and $B_4(t)=\lambda\rho\int_0^\ell s y_t y_{st} ds$, integrating by parts, one gets

$$
B_2(t)=\frac{\lambda\rho\ell}{2}x_t^2(\ell,t)-\frac{\lambda\rho}{2}\int_0^\ell x_t^2 ds \tag{26}
$$

$$
B_4(t)=\frac{\lambda\rho\ell}{2}y_t^2(\ell,t)-\frac{\lambda\rho}{2}\int_0^\ell y_t^2 ds. \tag{27}
$$

For $B_3(t)=\lambda\rho\int_0^\ell s y_{tt} y_s ds$, from the governing equation (2), it has

$$
\begin{aligned}
B_3(t)=&\lambda EA\int_0^\ell s y_s y_{ss} ds+\lambda EA\int_0^\ell s y_s x_s x_{ss} ds \\
&+\lambda\int_0^\ell s y_s f_y ds.
\end{aligned}
\tag{28}
$$

Further, one yields

$$
\begin{aligned}
B_3(t)=&\frac{\lambda EA\ell}{2}y_s^2(\ell,t)-\frac{\lambda EA}{2}\int_0^\ell y_s^2 ds-\frac{\lambda EA}{2}\int_0^\ell x_s^2 y_s ds \\
&+\frac{\lambda\ell EA}{2}x_s^2(\ell,t)y_s(\ell,t)-\frac{\lambda EA}{2}\int_0^\ell s x_s^2 y_{ss} ds \\
&+\lambda\int_0^\ell s y_s f_y ds.
\end{aligned}
\tag{29}
$$

Substituting (25)–(27) and (29) into (23), one has

$$
\begin{aligned}
\Gamma_{3t}(t)\le&\frac{\lambda\ell T_0(\ell)}{2}x_s^2(\ell,t)+\frac{\lambda\ell EA}{2}y_s^2(\ell,t)+\frac{\lambda\rho\ell}{2}x_t^2(\ell,t) \\
&+\left[\frac{3\lambda\ell\kappa(\ell)}{2}+\frac{3\lambda\ell EA}{8}\right]x_s^4(\ell,t)+\frac{\lambda\rho\ell}{2}y_t^2(\ell,t) \\
&+\lambda\ell EA x_s^2(\ell,t)y_s(\ell,t)-\frac{\lambda\rho}{2}\int_0^\ell x_t^2 ds-\frac{\lambda\rho}{2}\int_0^\ell y_t^2 ds \\
&-\frac{\lambda}{2}\int_0^\ell [T_0(s)-T_{0s}(s)-\lambda\ell\sigma_4]x_s^2 ds+\frac{\lambda\ell}{\sigma_4}\int_0^\ell f_x^2 ds \\
&-\int_0^\ell\left[\frac{3\lambda}{2}\kappa(s)-\frac{\lambda}{2}s\kappa_s(s)+\frac{3\lambda EA}{8}-\lambda EA\sigma_3\right]x_s^4 ds \\
&-\left(\frac{\lambda EA}{2}-\frac{\lambda EA}{\sigma_3}-\lambda\ell\sigma_5\right)\int_0^\ell y_s^2 ds+\frac{\lambda\ell}{\sigma_5}\int_0^\ell f_y^2 ds
\end{aligned}
\tag{30}
$$

where $\sigma_3,\sigma_4,\sigma_5>0$ are constants.

The time derivative of $\Gamma_4(t)$ is

$$
\begin{aligned}
\Gamma_{4t}(t)=&-\gamma_3\widetilde{T_0}(\ell,t)\widehat{T_{0t}}(\ell,t)-\gamma_4\widetilde{m}(t)\widehat{m}_t(t) \\
&-\gamma_5\widetilde{EA}(t)\widehat{EA}_t(t).
\end{aligned}
\tag{31}
$$

Substituting the adaptive laws (13)−(15) into (31), one obtains

$$
\begin{aligned}
\Gamma_{4t}(t)=&a\widetilde{T_0}(\ell,t)x_s(\ell,t)\zeta(t)\ln\frac{2b^2}{b^2-x_s^2(\ell,t)} \\
&-a\widetilde{T_0}(\ell,t)\zeta(t)x_s(\ell,t)+\phi_1\widetilde{T_0}(\ell,t)\widehat{T_0}(\ell,t) \\
&-a\widetilde{m}(t)\zeta^2(t)\frac{x_s(\ell,t)x_{st}(\ell,t)}{b^2-x_s^2(\ell,t)}+\phi_2\widetilde{m}(t)\widehat{m}(t) \\
&-a\widetilde{m}(t)x_{st}(\ell,t)\zeta(t)\ln\frac{2b^2}{b^2-x_s^2(\ell,t)}+a\widetilde{EA}(t)\cdot \\
&\left[\frac{x_s^3(\ell,t)}{2}+x_s(\ell,t)y_s(\ell,t)\right]\zeta(t)\ln\frac{2b^2}{b^2-x_s^2(\ell,t)} \\
&-a\zeta(t)\widetilde{EA}(t)\left[\frac{x_s^3(\ell,t)}{2}+x_s(\ell,t)y_s(\ell,t)\right] \\
&-a\widetilde{m}(t)\eta(t)y_{st}(\ell,t)+\phi_3\widetilde{EA}(t)\widehat{EA}(t).
\end{aligned}
\tag{32}
$$

Further, utilizing Young's inequality, one gets

$$
\begin{aligned}
\Gamma_{4t}(t)\le&a\zeta(t)\Bigg[\widetilde{T_0}(\ell,t)x_s(\ell,t)+\widetilde{EA}(t)x_s(\ell,t)y_s(\ell,t) \\
&+\frac{\widetilde{EA}(t)}{2}x_s^3(\ell,t)\Bigg]\ln\frac{2b^2}{b^2-x_s^2(\ell,t)}-\frac{\phi_1}{2}\widetilde{T_0}^2(\ell,t) \\
&-a\widetilde{m}(t)x_{st}(\ell,t)\zeta(t)\ln\frac{2b^2}{b^2-x_s^2(\ell,t)}-\frac{\phi_2}{2}\widetilde{m}^2(t) \\
&-a\widetilde{m}(t)\zeta^2(t)\frac{x_s(\ell,t)x_{st}(\ell,t)}{b^2-x_s^2(\ell,t)}-\frac{\phi_3}{2}\widetilde{EA}^2(t) \\
&-a\zeta(t)\Bigg[\widetilde{T_0}(\ell,t)x_s(\ell,t)+\widetilde{EA}(t)x_s(\ell,t)y_s(\ell,t) \\
&+\frac{\widetilde{EA}(t)}{2}x_s^3(\ell,t)\Bigg]-a\widetilde{m}(t)\eta(t)y_{st}(\ell,t) \\
&+\frac{\phi_1}{2}T_0^2(\ell)+\frac{\phi_2}{2}m^2+\frac{\phi_3}{2}EA^2.
\end{aligned}
\tag{33}
$$

According to (16), it is obtained that

$$
\Gamma_t(t)=\Gamma_{1t}(t)+\Gamma_{2t}(t)+\Gamma_{3t}(t)+\Gamma_{4t}(t) \tag{34}
$$

With the combination of (18), (22), (30), and (33), $\Gamma_t(t)$ becomes

$$
\begin{aligned}
\Gamma_t(t)\le&-ak_1\zeta^2(t)\ln\frac{2b^2}{b^2-x_s^2(\ell,t)}-\left(\frac{ak_2}{2}-\frac{\lambda\ell\rho}{2}\right)x_t^2(\ell,t) \\
&-\left(aT_0(\ell)-\frac{\lambda\ell T_0(\ell)}{2}-\frac{ak_2}{2}\right)x_s^2(\ell,t)-ak_3\eta^2(t) \\
&-\int_0^\ell\left[\frac{3\lambda}{2}\kappa(s)-\frac{\lambda}{2}s\kappa_s(s)+\frac{3\lambda EA}{8}-\lambda EA\sigma_3\right]x_s^4 ds \\
&-\frac{\lambda}{2}\int_0^\ell [T_0(s)-T_{0s}(s)-\lambda\ell\sigma_4]x_s^2 ds+\frac{a\gamma_1}{2}d_x^2(t) \\
&-\left(aEA-\frac{\lambda\ell EA}{2}-\frac{\left|\frac{3aEA}{2}-\lambda\ell EA\right|}{\sigma_6}-\frac{ak_4}{2}\right)y_s^2(\ell,t) \\
&-\Bigg[2a\kappa(\ell)+\frac{aEA}{2}-\frac{3\lambda\ell\kappa(\ell)}{2}-\left|\frac{3aEA}{2}-\lambda\ell EA\right|\sigma_6 \\
&-\frac{3\lambda\ell EA}{8}\Bigg]x_s^4(\ell,t)-\left(\frac{\lambda\rho}{2}-a\sigma_1\right)\int_0^\ell x_t^2 ds \\
&-\left(\frac{\lambda EA}{2}-\frac{\lambda EA}{\sigma_3}-\lambda\ell\sigma_5\right)\int_0^\ell y_s^2 ds+\frac{a\gamma_2}{2}d_y^2(t) \\
&-\left(\frac{\lambda\rho}{2}-a\sigma_2\right)\int_0^\ell y_t^2 ds+\left(\frac{a}{\sigma_1}+\frac{\lambda\ell}{\sigma_4}\right)\int_0^\ell f_x^2 ds \\
&+\left(\frac{a}{\sigma_2}+\frac{\lambda\ell}{\sigma_5}\right)\int_0^\ell f_y^2 ds-\left(\frac{a\gamma_1}{2}-\frac{a\sigma_3}{\delta_1}\right)\widetilde{d_x}^2(t) \\
&-\left(\frac{a\gamma_2}{2}-\frac{a\sigma_4}{\delta_2}\right)\widetilde{d_y}^2(t)+\frac{a}{\delta_1\sigma_3}d_{xt}^2(t)+\frac{a}{\delta_2\sigma_4}d_{yt}^2(t) \\
&-\frac{\phi_1}{2}\widetilde{T_0}^2(\ell,t)-\frac{\phi_2}{2}\widetilde{m}^2(t)-\frac{\phi_3}{2}\widetilde{EA}^2(t)+\frac{\phi_1}{2}T_0^2(\ell) \\
&+\frac{\phi_2}{2}m^2+\frac{\phi_3}{2}EA^2
\end{aligned}
\tag{35}
$$

where $\sigma_6>0$ is a constant and a, λ, ρ, k_2, k_4, σ_i, for $i=1,\ldots,5$ are selected to satisfy the following conditions:

$$
ak_2-\lambda\ell\rho>0 \tag{36}
$$

$$
ak_4-\lambda\ell\rho>0 \tag{37}
$$

$$
aT_0(\ell)-\frac{\lambda\ell T_0(\ell)}{2}-\frac{ak_2}{2}>0 \tag{38}
$$

$$
aEA-\frac{\lambda\ell EA}{2}-\frac{\left|\frac{3aEA}{2}-\lambda\ell EA\right|}{\sigma_6}-\frac{ak_4}{2}>0 \tag{39}
$$

$$
\begin{aligned}
&2a\kappa(\ell)+\frac{aEA}{2}-\frac{3\lambda\ell\kappa(\ell)}{2}-\frac{3\lambda\ell EA}{8} \\
&-\left|\frac{3aEA}{2}-\lambda\ell EA\right|\sigma_6>0
\end{aligned}
\tag{40}
$$

$$\tau_1 = \frac{\lambda\rho}{2} - a\sigma_1 > 0 \tag{41}$$

$$\tau_2 = \frac{\lambda\rho}{2} - a\sigma_2 > 0 \tag{42}$$

$$\tau_3 = T_0(s) - T_{0s}(s) - \lambda\ell\sigma_4 > 0 \tag{43}$$

$$\tau_4 = \frac{\lambda EA}{2} - \frac{\lambda EA}{\sigma_3} - \lambda\ell\sigma_5 > 0 \tag{44}$$

$$\tau_5 = \frac{3\lambda}{2}\kappa(s) + \lambda s\kappa_s(s) + \frac{3\lambda EA}{8} - \lambda EA\sigma_3 > 0 \tag{45}$$

$$\tau_6 = \frac{a\gamma_1}{2} - \frac{a\sigma_3}{\delta_1} > 0 \tag{46}$$

$$\tau_7 = \frac{a\gamma_2}{2} - \frac{a\sigma_4}{\delta_2} \tag{47}$$

$$\begin{aligned} 0 < \varepsilon = &\left(\frac{a}{\sigma_1} + \frac{\lambda\ell}{\sigma_4}\right)\ell f_1^2 + \left(\frac{a}{\sigma_2} + \frac{\lambda\ell}{\sigma_5}\right)\ell f_2^2 + \frac{a\gamma_1}{2}d_1^2 \\ &+ \frac{a\gamma_2}{2}d_2^2 + \frac{a}{\delta_1\sigma_3}d_3^2 + \frac{a}{\delta_2\sigma_4}d_4^2 + \frac{\phi_1}{2}T_0^2(\ell) \\ &+ \frac{\phi_2}{2}m^2 + \frac{\phi_3}{2}EA^2 < +\infty. \end{aligned} \tag{48}$$

Applying (36)–(48) on (35), it follows that:

$$\begin{aligned} \Gamma_t(t) \le &-\tau_1\int_0^\ell x_t^2 ds - \tau_2\int_0^\ell y_t^2 ds - \tau_3\int_0^\ell x_s^2 ds - \tau_4\int_0^\ell y_s^2 ds \\ &-\tau_5\int_0^\ell x_s^4 ds - ak_1\zeta^2(t)\ln\frac{2b^2}{b^2 - x_s^2(\ell,t)} - ak_3\eta^2(t) \\ &-\left(\frac{a\gamma_1}{2} - \frac{a\sigma_3}{\delta_1}\right)\widetilde{d_x}^2(t) - \left(\frac{a\gamma_2}{2} - \frac{a\sigma_4}{\delta_2}\right)\widetilde{d_y}^2(t) \\ &-\frac{\phi_1}{2}\widetilde{T_0}^2(\ell,t) - \frac{\phi_2}{2}\widetilde{m}^2(t) - \frac{\phi_3}{2}\widetilde{EA}^2(t) + \varepsilon \\ \le &-\alpha_3\left[\Pi(t) + \Gamma_2(t) + \widetilde{T_0}^2(\ell,t) + \widetilde{m}^2(t) + \widetilde{EA}^2(t)\right] + \varepsilon \end{aligned} \tag{49}$$

where $\alpha_3 = \min\{\tau_1,\ \tau_2,\ \tau_3,\ \tau_4,\ \tau_5,\ 2k_1/m,\ 2k_3/m,\ 2\delta_1\tau_6/a,\ 2\delta_2\tau_7/a,\ \phi_1/2, \phi_3/2\}$. Thus, together with (6) and (49), one further obtains

$$\Gamma_t(t) \le -\alpha\Gamma(t) + \varepsilon \tag{50}$$

where $\alpha = \alpha_3/\alpha_2 > 0$. ■

VI. Proof of Theorem 1

Theorem 1: For coupled strings represented by (1), (2) and boundary conditions (3)–(5), under Assumption 1 and adaptive control protocols (9), (10) with disturbance observers (11), (12) and adaptive laws (13)–(15), we can conclude that all signals of the system are bounded, the boundary tension of the string satisfies the constraint $|T(\ell,t)| < T_M$, $\forall t \in [0,\infty)$, the boundary disturbances are effectively restrained and the closed-loop system is uniformly bounded stable.

Proof: Multiplying (15) by $e^{\alpha t}$ gives

$$\Gamma_t(t)e^{\alpha t} \le -\alpha\Gamma(t)e^{\alpha t} + \varepsilon e^{\alpha t}. \tag{51}$$

The above equation is equivalent to

$$\frac{\partial}{\partial t}\left(\Gamma(t)e^{\alpha t}\right) \le \varepsilon e^{\alpha t}. \tag{52}$$

Then, integrating (52) with respect to t from 0 to t

$$\Gamma(t) \le \left(\Gamma(0) - \frac{\varepsilon}{\alpha}\right)e^{-\alpha t} + \frac{\varepsilon}{\alpha} \le \Gamma(0)e^{-\alpha t} + \frac{\varepsilon}{\alpha} \tag{53}$$

which infers $\Gamma(t)$ is bounded. Applying Young's inequality and combining (17) with (21), we obtain

$$\begin{aligned} \frac{a\underline{T_0}}{2\ell}x^2(s,t) &\le \frac{a}{2}\int_0^\ell T_0(s)x_s^2(x,t)ds \\ &\le \Gamma_1(t) \le \Gamma_1(t) + \Gamma_2(t) \le \frac{1}{\alpha_1}\Gamma(t) \end{aligned} \tag{54}$$

$$\begin{aligned} \frac{aEA}{2\ell}y^2(s,t) &\le \frac{aEA}{2}\int_0^\ell y_s^2(x,t)ds \\ &\le \Gamma_1(t) \le \Gamma_1(t) + \Gamma_2(t) \le \frac{1}{\alpha_1}\Gamma(t). \end{aligned} \tag{55}$$

Rearrange the terms in the above inequality appropriately. It is obtained that $x(s,t)$ and $y(s,t)$ are bounded, that is

$$|x(s,t)| \le \sqrt{\frac{2\ell}{a\alpha_1\underline{T_0}}\left(V(0) + \frac{\varepsilon}{\alpha}\right)} \tag{56}$$

$$|y(s,t)| \le \sqrt{\frac{2\ell}{a\alpha_1 EA}\left(V(0) + \frac{\varepsilon}{\alpha}\right)} \tag{57}$$

$\forall(s,t) \in [0,\ell]\times[0,\infty)$. Further, we derive

$$\lim_{t\to\infty}|x(s,t)| \le \sqrt{\frac{2\ell\varepsilon}{aT\alpha_1\alpha}} \tag{58}$$

$$\lim_{t\to\infty}|y(s,t)| \le \sqrt{\frac{2\ell\varepsilon}{aEA\alpha_1\alpha}} \tag{59}$$

$\forall(s,t) \in [0,\ell]\times[0,\infty)$.

From the two inequalities (54), (55), we see that $\Gamma_1(t)$ is bounded $\forall t \in [0,\infty)$. Since $\Gamma_1(t)$ is bounded, $x_t(s,t)$, $x_s(s,t)$, $y_t(s,t)$ and $y_s(s,t)$ are bounded $\forall(s,t) \in [0,\ell] \times[0,\infty)$. From (1), the kinetic energy of the system is bounded, it follows that $x_{st}(s,t)$ and $y_{st}(s,t)$ are also bounded by utilizing property 1. Similarly, it follows from (2) and property 1 that $x_{ss}(s,t)$ and $y_{ss}(s,t)$ are also bounded. Then, utilizing Assumption 1, system governing equations, by means of boundary conditions and the above analysis, it is easy to obtain that $x_{tt}(s,t)$ and $y_{tt}(s,t)$ are also bounded. In addition, From (53), we have parameter estimation errors $\widetilde{T_0}(\ell,t)$, $\widetilde{m}(t)$, and $\widetilde{EA}(t)$ are bounded. Thus, $\widehat{T_0}(\ell,t)$, $\widehat{m}(t)$, and $\widehat{EA}(t)$ are bounded. In conclusion, adaptive boundary control controllers $U_x(t)$ and $U_y(t)$ we designed are bounded. In summary, the proposed two adaptive boundary controllers $U_x(t)$ and $U_y(t)$ guarantee that all signals in the closed-loop system are bounded.

From the definition of $\Gamma_2(t)$, it is easy to see that $\Gamma_2(t) \to \infty$ when $|x_s(\ell,t)| \to b$. By (53), we know that $\Gamma_2(t)$ is bounded, so $|x_s(s,t)| \ne b$. Considering $-b < x_s(\ell,0) < b$, we further deduce that $-b < x_s(\ell,t) < b$, $\forall t \in [0,\infty)$. Together with the tension expression, it is clear that $-T_M < T(\ell,t) < T_M$ holds on $\forall t \in [0,\infty)$, so the boundary tension $T(\ell,t)$ satisfies the constraint. ■

VII. Four Remarks

Remark 1: The main tool utilized in this paper is the estimation-based adaptive constraint control method. Since this paper considers the case where the system has unknown boundary perturbations and unknown system parameters, two disturbance observers and three parameter adaptive laws are designed to estimate the unknown boundary perturbations $d_x(t)$, $d_y(t)$ and unknown parameters $T_0(\ell)$, m, $\boldsymbol{EA}$, respectively. In addition, the boundary tension constraint problem of the system is also considered and the logarithmic BLF is selected to deal with it. Therefore, this paper effectively solves the vibration suppression problem of the transverse-longitudinal coupled string system using the estimation-based adaptive constraint control method.

Remark 2: In this paper, the unknown parameters $T_0(\ell)$, m and EA are considered. To solve this problem, the modification terms ϕ_1, ϕ_2, and ϕ_3 in (13)–(15) are introduced to improve the robustness of the closed-loop system, which are used to regulate $\widehat{T_0}(\ell,t)$, $\widehat{m}(t)$, $\widehat{EA}(t)$, respectively, to avoid their fluctuation to very large values that may affect the control scheme.

Remark 3: From the above analysis, it can be seen that the system states $x(s,t)$ and $y(s,t)$ can be arbitrarily small as long as the design control parameters are properly chosen. According to the expression of α_3, it is clear that the increase of control gains k_1, k_3 may cause α_3 to increase. Then, the value of α will increase, which eventually makes $\sqrt{2\ell\varepsilon/aT\alpha_1\alpha}$ and $\sqrt{2\ell\varepsilon/aEA\alpha_1\alpha}$ decrease, i.e., it can give a better vibration reduction performance. But, it will generate a high gain control scheme by increasing k_1, k_3. Therefore, in practical engineering, to achieve good vibration reduction performance and to obtain an optimized control scheme, the design parameters should be carefully adapted.

Remark 4: All signals of the adaptive boundary controllers (9) and (10) can be obtained by backward difference algorithm or by sensor measurements. $x(\ell,t)$ and $y(\ell,t)$ are measured by laser displacement sensor at the boundary of the string and $x_s(\ell,t)$ and $y_s(\ell,t)$ are obtained by inclinometer. Furthermore, $x_t(\ell,t)$, $y_t(\ell,t)$, $x_{st}(\ell,t)$, and $y_{st}(\ell,t)$ are calculated using the backward difference algorithm for $x(\ell,t)$, $y(\ell,t)$, $x_s(\ell,t)$, $y_s(\ell,t)$, respectively.